

# BINOMIAL REGULAR SEQUENCES AND FREE SUMS

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**ABSTRACT.** Recently several authors have proved results on Ehrhart series of free sums of rational polytopes. In this note we treat these results from an algebraic viewpoint. Instead of attacking combinatorial statements directly, we derive them from structural results on affine monoids and their algebras that allow conclusions for Hilbert and Ehrhart series. We characterize when a binomial regular sequence generates a prime ideal or even normality is preserved for the residue class ring.

## 1. INTRODUCTION

Recently several authors have proved results on Ehrhart series of free sums of rational polytopes; see Beck and Hoşten [1], Braun [5] and Beck, Jayawant, and McAllister [2]. In this note we treat these results from an algebraic viewpoint. Instead of attacking combinatorial statements directly, we derive them from structural results on affine monoids and their algebras that allow conclusions for Hilbert and Ehrhart series. This procedure follows the spirit of the monograph [6] to which the reader is referred for affine monoids and their algebras.

Our approach is best explained by the motivating example, namely free sums of rational polytopes and their Ehrhart series. The *Ehrhart series* of a rational polytope  $P$  is the (formal) power series  $E_P = \sum_{k=0}^{\infty} E(P, k)t^k$  where  $E(P, k)$  counts the lattice points in the homothetic multiple  $kP$ ; see Beck and Robbins [3] for a gentle introduction to the fascinating area of Ehrhart series.

One says that  $R = \text{conv}(P \cup Q)$  is the *free sum* of the rational polytopes  $P$  and  $Q$  if  $0 \in P \cap Q$ , the vector subspaces  $\mathbb{R}P$  and  $\mathbb{R}Q$  intersect only in 0, and

$$(\mathbb{Z}^m \cap \mathbb{R}R) = (\mathbb{Z}^m \cap \mathbb{R}P) + (\mathbb{Z}^m \cap \mathbb{R}Q).$$

It has been proved in [2, Theorem 1.4] that the Ehrhart series of the three polytopes are related by the equation

$$E_R = (1 - T)E_P E_Q \quad (*)$$

if and only if at least one of the polytopes  $P$  and  $Q$  is described by inequalities of type  $a_1x_1 + \dots + a_mx_m \leq b$  with  $a_1, \dots, a_n \in \mathbb{Z}$  and  $b \in \{0, 1\}$ .

We approach the validity of equation  $(*)$  by considering the *Ehrhart monoid*

$$\mathcal{E}(P) = \{(x, k) : x \in kP \cap \mathbb{Z}^m\} = \mathbb{R}_+(P \times \{1\}) \cap \mathbb{Z}^{m+1}.$$

The Ehrhart series is the Hilbert series of  $\mathcal{E}(P)$  or, equivalently, of the monoid algebra  $K[\mathcal{E}(P)]$  over a field  $K$ , and therefore standard techniques for computing Hilbert series can be applied. Ehrhart monoids are normal: if  $nx \in \mathcal{E}(P)$  for some  $x$  in the group  $\mathbb{Z}\mathcal{E}(P)$  and  $n \in \mathbb{Z}_+$ ,  $n > 0$ , then  $x \in \mathcal{E}(P)$ . The normality of a monoid  $M$  is equivalent to the normality of  $K[M]$ .

The free sum arises from the free join by a projection along the line through the representatives of the origins in  $P$  and  $Q$ , respectively, in the free join. The algebraic counterpart of the projection is the passage from the direct sum  $\mathcal{E}(P) \oplus \mathcal{E}(Q)$  to a quotient  $M$ . By Corollary 2.5,  $M$  is automatically an affine monoid in this situation. However, the crucial question is whether  $M$  is naturally isomorphic to  $\mathcal{E}(R)$ , and this is the case if and only if  $M$  is normal. In terms of monoid algebras, the quotient is given by residue classes modulo a binomial. Therefore the validity of  $(*)$  can be seen as a special case of the preservation of normality modulo a binomial in a normal monoid algebra, for which Theorem 3.3 provides a necessary and sufficient condition.

In [2, Corollary 5.8] the intersection of  $\mathbb{R}P$  and  $\mathbb{R}Q$  in 0 has been generalized to the intersection of the affine hulls  $\text{aff}(P)$  and  $\text{aff}(Q)$  in a single rational point  $z \in P \cap Q$ , and the corresponding generalization of  $(*)$  follows by entirely the same argument (Corollary 3.7).

Our discussion above shows that it is worthwhile to characterize when a binomial (or more generally a regular sequence of binomials) in an affine monoid domain generates a prime ideal (Theorem 2.1 and Corollary 2.3), or when even normality is preserved modulo such a binomial (Theorem 3.3 and Corollary 3.4). Also the main reduction step in Bruns and Römer [9] is of this type.

It would be possible to mold the results of this note in the language of monoids and congruences, but the ring-theoretic environment is much richer in notions and methods, and results like Hochster's theorem on the Cohen-Macaulay property of normal affine monoid domains could hardly be formulated in pure monoid theory.

This work was initiated by discussions with Serkan Hoşten about [1] and then driven by the desire to prove the results of [2] and [5] in an algebraic way. We are grateful to Matthias Beck for directing our attention to these papers, and we thank Benjamin Braun, Serkan Hoşten, Tyrrell McAllister and Matteo Varbaro for their careful reading of a preliminary version and valuable suggestions.

## 2. INTEGRALITY

An *affine monoid* is a finitely generated submonoid of a group  $\mathbb{Z}^m$ . It is *positive* if  $x, -x \in M$  implies  $x = 0$ . For a field  $K$  the monoid algebra  $K[M]$  is a finitely generated  $K$ -subalgebra of the Laurent polynomial ring  $K[\mathbb{Z}^m]$ . We write  $X^x$  for the (Laurent) monomial with exponent vector  $x$ . Since the subgroup  $\text{gp}(M)$  of  $\mathbb{Z}^m$  generated by  $M$  is isomorphic to  $\mathbb{Z}^d$  for  $d = \text{rank } M = \text{rank } \text{gp}(M)$ , the subalgebra  $K[\text{gp}(M)] \subset K[\mathbb{Z}^m]$  is a Laurent polynomial ring in its own right. For an extensive treatment of affine monoids and their algebras we refer the reader to Bruns and Gubeladze [6], in particular to Chapter 4.

A *(multi)grading* on a monoid  $M$  is a  $\mathbb{Z}$ -linear map  $\deg : \text{gp}(M) \rightarrow \mathbb{Z}^d$  for some  $d > 0$ . If the Hilbert function  $H(M, g) = \#\{x \in M : \deg x = g\}$  is *finite* for all  $g$ , we can define the Hilbert series

$$H_M(T) = \sum_{g \in \mathbb{Z}^d} H(M, g) T^g$$

where  $T$  stands for indeterminates  $T_1, \dots, T_d$  and  $T^g = T_1^{g_1} \dots T_d^{g_d}$ . See [6, Ch. 6] for the basic theorems on Hilbert series. A priori,  $H_M(T)$  lives in the  $\mathbb{Z}[T_1, \dots, T_d]$ -module  $\mathbb{Z}[[T_1, \dots, T_d]]$  of formal Laurent series.

Every grading on  $M$  is the specialization of the *fine grading* in which  $\deg$  is simply the given embedding  $\text{gp}(M) \hookrightarrow \mathbb{Z}^n$ . We denote the Hilbert series of the fine grading by  $\mathbb{H}_M$ . Since  $M$  can be recovered from  $\mathbb{H}_M$ , it is justified to call it the generating function of  $M$ .

We say that  $M$  is *positively (multi)graded* if  $\deg(M)$  is a positive submonoid of  $\mathbb{Z}^d$  and the elements of  $K \subset K[M]$  are the only ones of degree 0. This implies the finiteness of the Hilbert function. By the classical theorem of Hilbert-Serre,  $H_M(T)$  is the Laurent series expansion of a rational function (with respect to the positive submonoid  $\deg(M)$ ).

A few more pieces of terminology and notation: we say that a nonzero  $x \in \mathbb{Z}^n$  is *unimodular* if  $x$  generates a direct summand. The cone generated by  $A \subset \mathbb{R}^n$  is denoted by  $\text{cone}(A)$ , and  $\text{aff}(A)$  is the affine subspace spanned by  $A$ .

For the basic theory of zerodivisors,  $R$ -sequences and depth in Noetherian rings we refer the reader to Bruns and Herzog [7].

**Theorem 2.1.** *Let  $K$  be a field,  $M$  an affine monoid, and  $x, y \in M$  noninvertible,  $x \neq y$ . Then the following statements (1) and (2) are equivalent:*

- (1)  $X^x - X^y$  generates a prime ideal in  $K[M]$ .
- (2) (a)  $X^x, X^y$  is a  $K[M]$ -sequence;  
(b)  $\text{gp}(M)/\mathbb{Z}(x - y)$  is torsionfree.

Moreover, if  $\phi : M \rightarrow M'$  is a surjective homomorphism onto an affine monoid  $M'$  with  $\text{rank } M' = \text{rank } M - 1$  and  $\phi(x) = \phi(y)$ , then (1) and (2) are equivalent to

- (3)  $K[M'] = K[M]/(X^x - X^y)$  under the induced homomorphism.

Finally, if in this situation  $M'$  is positively multigraded and  $\phi(z) \neq 0$  for all nonzero  $z \in M$ , then (1), (2), and (3) are equivalent to

- (4)  $H_{M'} = (1 - T^g)H_M$  with respect to the induced grading on  $M$ ,  $g = \deg \phi(x)$ .

*Proof.* Let us start with the implication (2)  $\implies$  (1). First we prove that no monomial is a zerodivisor modulo  $X^x - X^y$  if (2)(a) holds. In fact, suppose that  $X^z$  is such a zerodivisor. Then it is contained in an associated prime ideal  $P$  of  $X^x - X^y$ . But  $P$  is an associated prime ideal of any nonzero element of  $R = K[M]$  it contains (since  $R$  is an integral domain). Therefore  $P$  is an associated prime ideal of  $X^z$  as well. Associated prime ideals of monomials are generated by monomials [6, 4.9], and so  $P$  contains both  $X^x$  and  $X^y$  together with  $X^x - X^y$ . This is a contradiction since  $X^x, X^y$  is a regular sequence in the localization  $R_P$ .

It follows that  $(X^x - X^y)$  is the contraction of its extension to the Laurent polynomial ring  $K[\text{gp}(M)]$  (see [6, 4.C]). So it is enough that  $(X^x - X^y)K[\text{gp}(M)]$  is a prime ideal. This follows from (2)(b) since  $(X^x - X^y)K[\text{gp}(M)] = (1 - X^{y-x})K[\text{gp}(M)]$  and  $X^{y-x}$  is an indeterminate in  $K[\text{gp}(M)]$  after a suitable choice of a basis of  $\text{gp}(M)$ .

For the converse we first derive (2)(a). If  $(X^x - X^y)$  is a prime ideal, then no monomial can be a zerodivisor modulo  $X^x - X^y$ . On the other hand, if  $X^y$  were a zerodivisor modulo  $X^x$ , then it would be contained in an associated prime ideal  $P$  of  $X^x$ . But such  $P$  is monomial and also an associated prime ideal of  $X^x - X^y$ . Thus it would be equal to  $(X^x - X^y)$ , which is not monomial.

(2)(b) follows from (1) since the primeness of the extension of  $(X^x - X^y)$  to  $K[\text{gp}(M)]$  evidently implies that  $\text{gp}(M)/\mathbb{Z}(x - y)$  is torsionfree [6, 4.32]; also see Remark 2.2(c).

For the equivalence of (3) to (1) and (2) we note that the natural surjection from  $K[M]$  to  $K[M']$  factors through  $K[M]/(X^x - X^y)$ . Since  $\text{rank } M' = \text{rank } M - 1$ , one has  $\dim K[M'] = \dim K[M] - 1$  (we consider Krull dimension here), and  $\text{Ker } \varphi$  is a height 1 prime ideal. So the natural isomorphism  $K[M'] = K[M]/\text{Ker } \varphi$  turns into  $K[M'] = K[M]/(X^x - X^y)$  if and only if  $\text{Ker } \varphi = (X^x - X^y)$ .

For statement (4) to make sense, we need that  $\varphi(z) \neq 0$  for all  $z \in M$ . This assumption implies that we indeed obtain a multigrading on  $M$  by setting  $\deg z = \deg \varphi(z)$ . The equivalence of (4) follows by the same argument: one has

$$H_{K[M]/(X^x - X^y)} = (1 - T^g)H_{K[M]}$$

since  $X^x - X^y$  is homogeneous of degree  $g$ , and  $H_{K[M']} = (1 - T^g)H_{K[M]}$  if and only if the two algebras are isomorphic.  $\square$

**Remark 2.2.** (a) Condition (3) has been formulated in view of the applications below. If (1) holds, then  $K[M]/(X^x - X^y)$  is automatically an affine monoid domain  $K[M']$  whose underlying monoid is the image of  $M$  in  $\text{gp}(M)/\mathbb{Z}(x - y)$  [6, 4.32].

(b) It is not hard to see that monomials  $X^{x_1}, \dots, X^{x_n}$  form a  $K[M]$ -sequence if and only if  $X^{x_i}, X^{x_j}$  is a  $K[M]$ -sequence for all  $i \neq j$ . Nevertheless condition (2)(a) is not easy to check in general. If  $K[M]$  is Cohen-Macaulay or satisfies at least Serre's condition  $(S_2)$ , for example if  $M$  is normal, then (2)(a) is equivalent to the fact that there is no facet  $F$  of  $\text{cone}(M)$  with  $x, y \notin F$ , or, in other words, every facet contains at least one of  $x$  or  $y$ . Indeed, in a ring satisfying  $(S_2)$  the associated prime ideals of non-zerodivisors have height 1, and the height 1 monomial prime ideals are exactly those spanned by the monomials  $X^z, z \notin F$ , for some facet  $F$  of  $\text{cone}(M)$  [6, 4.D].

(c) One should note that  $K[\mathbb{Z}^d]/I$  is not only a domain, but even a regular domain if  $I$  is generated by binomials  $X^{x_1} - X^{y_1}, \dots, X^{x_n} - X^{y_n}$  such that  $x_1 - y_1, \dots, x_n - y_n$  generate a rank  $n$  direct summand. By induction it is enough to prove the claim for  $n = 1$ ,  $x = x_1, y = y_1$ . With respect to a basis of  $\mathbb{Z}^d$  containing  $y - x$  as the first element,  $K[\mathbb{Z}^d] = K[Y_1^{\pm 1}, \dots, Y_d^{\pm 1}]$  with  $Y_1 = X^{y-x}$ , and  $K[\mathbb{Z}^d]/(1 - Y_1)$  arises from the regular domain  $K[Y_1, \dots, Y_d]/(1 - Y_1)$  by the inversion of the monomials in  $Y_1, \dots, Y_d$ .

For a finite subset  $A \subset \mathbb{Z}^m$  let the (automatically positive) *monoid*  $M(A)$  over  $A$  be the submonoid of  $\mathbb{Z}^{m+1}$  generated by the vectors  $(x, 1) \in \mathbb{Z}^{m+1}$ ,  $x \in A$ . This type of monoid will play a special role later on, but is useful already now for the construction of examples.

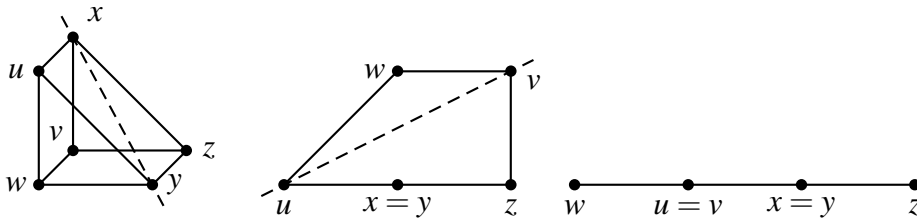


FIGURE 1. Successive identification of lattice points

The geometry behind Theorem 2.1 is illustrated by Figure 1. We start from the monoid  $M(A)$  where  $A$  is the set of vertices of the direct product of the unit 2-simplex and the unit 1-simplex. The monoid  $M'$  arising from the identification of  $x$  and  $y$  is then defined

by the 5 lattice points of the quadrangle in the middle, and if we further identify  $u$  and  $v$ , we end with the line segment on the right with its 4 lattice points. The polytopes in the middle and on the right are obtained from their left neighbors by projection along the line through the identified points, indicated by  $x = y$  and  $u = v$ .

We generalize the theorem to sequences of more than two elements, leaving the generalization of (3) and (4) to the reader.

**Corollary 2.3.** *With  $K$  and  $M$  as in Theorem 2.1, let  $x_1, \dots, x_n$ ,  $n \geq 2$ , be noninvertible elements of  $M$ . Then the following statements (1) and (2) are equivalent:*

- (1)  $X^{x_1} - X^{x_2}, \dots, X^{x_{n-1}} - X^{x_n}$  is a  $K[M]$ -sequence and generates a prime ideal  $P$ .
- (2) (a)  $X^{x_1}, \dots, X^{x_n}$  is a  $K[M]$ -sequence;  
 (b)  $x_1 - x_2, \dots, x_{n-1} - x_n$  generate a rank  $n - 1$  direct summand of  $\text{gp}(M)$ .

*Proof.* For the proof of the implication (1)  $\implies$  (2)(a) let  $Q$  be the prime ideal of  $K[M]$  generated by all noninvertible monomials. Since the associated prime ideals of monomial ideals are themselves monomial,  $X^{x_1}, \dots, X^{x_n}$  is a  $K[M]$ -sequence if and only if it is a  $K[M]_Q$ -sequence, and the latter property follows from  $\text{depth } K[M]_{Q'} \geq n$  for all prime ideals  $Q' \supset (X^{x_1}, \dots, X^{x_n})$  [7, 1.6.19].

A prime ideal  $Q' \supset (X^{x_1}, \dots, X^{x_n})$  contains the regular sequence  $X^{x_1} - X^{x_2}, \dots, X^{x_{n-1}} - X^{x_n}$  of length  $n - 1$  that generates the prime ideal  $P$ . Moreover  $Q$  contains  $X^{x_n}$  and  $\notin P$ . This implies  $\text{depth } K[M]_{Q'} \geq n$ , and (2)(a) has been verified. (2)(b) follows since the extension of  $P$  to  $K[\text{gp}(M)]$  is a prime ideal [6, 4.32].

For (2)  $\implies$  (1) we use induction for which the starting case  $n = 2$  is covered by the theorem. Let  $P' = (X^{x_1} - X^{x_2}, \dots, X^{x_{n-2}} - X^{x_{n-1}})$ ; by induction  $K[M]/P'$  is an affine monoid domain  $K[M']$  (see Remark 2.2(a)). The only critical condition is whether  $X^{x_{n-1}}, X^{x_n}$  is a  $K[M']$ -sequence since (2)(b) of the theorem is evidently satisfied. Let  $Q'$  be a prime ideal in  $K[M']$  containing  $X^{x_{n-1}}, X^{x_n}$ , and let  $Q$  be its preimage in  $K[M]$ . Then  $Q$  contains the total sequence  $X^{x_1}, \dots, X^{x_n}$ , and we conclude  $\text{depth } K[M]_Q \geq n$ . But modulo the regular sequence  $X^{x_1} - X^{x_2}, \dots, X^{x_{n-2}} - X^{x_{n-1}}$  of length  $n - 2$  the depth goes down by  $n - 2$ , and therefore  $\text{depth } K[M]_{Q'} \geq 2$ . This makes it impossible that  $X^n$  is a zerodivisor modulo  $X^{x_{n-1}}$  in  $K[M']$ .  $\square$

**Remark 2.4.** (a) For the proof of the implication (1)  $\implies$  (2)(a) of Corollary 2.3 we have only used that  $X^{x_1} - X^{x_2}, \dots, X^{x_{n-1}} - X^{x_n}, X^{x_n}$  is a  $K[M]$ -sequence. The converse does also hold.

Since  $(X^{x_1} - X^{x_2}, \dots, X^{x_{n-1}} - X^{x_n}, X^{x_n}) = (X^{x_1}, \dots, X^{x_n})$ , the same argument that has been used for (1)  $\implies$  (2)(a) shows that  $X^{x_1} - X^{x_2}, \dots, X^{x_{n-1}} - X^{x_n}, X^{x_n}$  is a  $K[M]_{Q'}$ -sequence. The only problem is to lift regularity of the sequence to  $K[M]$ . We can no longer use the fine grading, but it is sufficient that there is a multigrading for which (i)  $Q'$  is the ideal generated by the noninvertible homogeneous elements, and (ii)  $X^{x_1} - X^{x_2}, \dots, X^{x_{n-1}} - X^{x_n}, X^{x_n}$  are homogeneous. Then we are dealing with homogeneous elements in the  $*$ maximal ideal  $Q'$  of the  $*$ local ring  $K[M]$ . See [7, 1.5.15(c)] that covers the case of positive  $\mathbb{Z}$ -gradings; however, it is solely relevant that the grading group is torsionfree (Bourbaki [4, Ch. 4, § 3, no. 1]).

It remains to find a suitable grading. To this end we let  $U$  be the saturation of  $\mathbb{Z}(x_1 - x_2) + \cdots + \mathbb{Z}(x_{n-1} - x_n)$  in  $\text{gp}(M)$ . Then  $G = \text{gp}(M)/U$  is torsionfree, and the natural homomorphism  $\text{gp}(M) \rightarrow G$  is the right choice.

(b) We have assumed in Theorem 2.1 and in Corollary 2.3 that  $x_1, \dots, x_n$  are noninvertible. If one allows that one of the  $x_i$  is a unit in  $M$ , then (2)(a) makes no sense anymore since the definition of  $K[M]$ -sequence comprises the condition  $(x_1, \dots, x_n) \neq K[M]$ . But dropping this requirement and keeping only that  $x_i$  is not a zerodivisor modulo  $(x_1, \dots, x_{i-1})$  for  $i = 1, \dots, n$  is not the way out.

The ideal  $P$  generated by the  $X^{x_{i-1}} - X^{x_i}$  is independent of the order of the  $x_i$ , and especially its primeness does not depend on the order. However, the second property in (1), namely that the generators form a  $K[M]$ -sequence, may be order sensitive if one of the  $x_i$  is a unit and we have left the shelter of the “roof”  $Q'$  above. For a concrete example set  $M = M(A)$  where  $A$  is the set of the vertices 3-dimensional unit cube, and  $x, y, z$  are chosen as indicated in Figure 2. Then  $X^x - X^y, X^y - X^z, X^z - 1$ , corresponding to  $x, y, z, 0 \in M$ , is

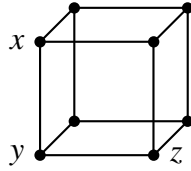


FIGURE 2. The unit cube

not a  $K[M]$ -sequence, although the permutation  $X^z - 1, X^y - X^z, X^x - X^y$ , corresponding to  $0, z, y, x \in M$ , is a  $K[M]$ -sequence, and both sequences generate the same prime ideal  $P$ . In fact,  $K[M]/P$  is isomorphic to the polynomial ring in one variable over  $K$ .

This is not really a surprise: in a non-local situation the fact that an ideal  $P$  is generated by a regular sequence of length 3 does not imply that every length 3 sequence generating  $P$  is regular.

The order that just made the generators of  $P$  a  $K[M]$ -sequence does always work: (2)(b) alone is equivalent to (1), provided  $x_1$  is a unit. Under this assumption all arguments remain essentially unchanged, except that the set of monomial ideals containing  $x_1, \dots, x_n$  is automatically empty.

(c) Binomial regular sequences in polynomial rings  $K[\mathbb{Z}_+^m]$  have been investigated in Fischer, Morris and Shapiro [10] and Fischer and Shapiro [11].

We now turn to a situation in which the conditions of Theorem 2.1 are automatically satisfied.

**Corollary 2.5.** *Let  $L, M$  and  $N$  be affine monoids,  $\varphi : M \oplus N \rightarrow L$  a surjective homomorphism with  $\text{rank } L = \text{rank } M + \text{rank } N - 1$ , and suppose that  $\varphi(x) = \varphi(y)$  for  $x \in M, y \in N, x \neq 0$  or  $y \neq 0$ . Furthermore assume that nonzero  $\varphi(z) \neq 0$  for all  $z \in M \oplus N$  and let  $L$  be positively multigraded such that  $\deg \varphi(x)$  is a unimodular element of the grading group.*

*Then  $K[L] \cong K[M \oplus N]/(X^x - X^y)$  and  $H_L = (1 - T^g)H_{M \oplus N}$  with respect to the grading on  $M \oplus N$  induced by the grading on  $L$ .*

*Proof.* We must verify the conditions (1)(a) and (b) of the theorem. For (a) the verification is a trivial exercise. For (b) let  $G$  be the grading group of  $L$ . The grading on  $L$  induces gradings on  $M$  and  $N$  via the embedding of  $M$  and  $N$ , respectively, into  $M \oplus N$ . Consider the homomorphism  $M \oplus N \rightarrow G \oplus G$ ,  $(u, v) \mapsto (\deg u, \deg v)$ . Under this homomorphism  $x - y = (x, -y)$  goes to the unimodular element  $(\deg x, -\deg y)$  of  $G \oplus G$ . Therefore  $x - y$  is unimodular in  $\text{gp}(M \oplus N)$ .  $\square$

As a special case of Corollary 2.5 we can consider the free sum of point configurations. Following [2] let  $A, B \subset \mathbb{R}^m$ . We say that  $A \cup B$  is the *free sum* of  $A$  and  $B$  if  $0 \in A \cap B$  and the vector subspaces  $\mathbb{R}A$  and  $\mathbb{R}B$  of  $\mathbb{R}^m$  intersect only in  $0$ . The relationship between  $M(A \cup B)$  and  $M(A) \oplus M(B)$  is given by part (1) of the next corollary in terms of monoid algebras.

**Corollary 2.6.** *Let  $A$  and  $B$  be finite subsets of  $\mathbb{Z}^m$  such that  $A \cup B$  is the free sum of  $A$  and  $B$ . Set  $x = (0, 1) \oplus 0$ ,  $y = 0 \oplus (0, 1)$ . Then*

- (1)  $K[M(A \cup B)] \cong K[M(A) \oplus M(B)] / (X^x - X^y)$ ;
- (2)  $\mathbb{H}_{M(A \cup B)} = (1 - T_{m+1}) \mathbb{H}_{M(A)} \mathbb{H}_{M(B)}$ .

*Proof.* We set  $M = M(A)$ ,  $N = M(B)$  and  $L = M(A \cup B)$ . Then the natural embeddings  $M \subset L$  and  $N \subset L$  induce a surjective homomorphism  $M \oplus N \rightarrow L$ ,  $(x, k) \oplus (y, l) \mapsto (x + y, k + l)$ . Both  $x$  and  $y$  go to  $(0, 1) \in L \subset \mathbb{Z}^{m+1}$ , and therefore to a unimodular element in  $\text{gp}(L)$ . It only remains to apply Corollary 2.5.  $\square$

**Remark 2.7.** We have formulated Corollary 2.6 for the fine grading. Since every other grading is a specialization of the fine grading, the formula in (2) holds for every coarser grading as well. In particular it holds for the *standard grading* on  $M(A)$ ,  $M(B)$  and  $M(A \cup B)$  in which  $\deg(x, k) = k \in \mathbb{Z}$ .

The formula in (2) was stated (for the standard grading) in [1, Lemma 10] without the factor  $1 - T_{m+1}$ . Therefore some of the results in [1] need an analogous correction, but this only concerns the denominators of the Hilbert series appearing there, and the statements about the numerator polynomials remain untouched.

The construction of free sums has been generalized in [2] as follows. We consider subsets  $A$  and  $B$  of  $\mathbb{R}^m$  such that  $\text{aff}(A)$  and  $\text{aff}(B)$  meet in a single point  $p_0$ ; see Figure 3

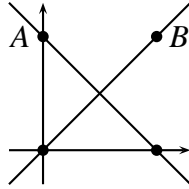


FIGURE 3. Intersection in a rational point

for a very simple example. In this example

$$H_{M(A \cup B)} = \frac{1 + T}{(1 - T)^3} = (1 - T^2) H_{M(A)} H_{M(B)}$$

in the standard grading. The “correction”  $1 - T^2$  reflects that  $(2p_0, 2) \in M(A) \cap M(B)$ .

Let  $A, B \subset \mathbb{Z}^m$  be finite and suppose that  $\text{aff}(A)$  and  $\text{aff}(B)$  meet exactly in  $p_0$ ; then necessarily  $p_0 \in \mathbb{Q}^m$ . We can form  $M(A)$ ,  $M(B)$  and  $M(A \cup B)$ . If  $p_0 \in \mathbb{Z}^m \cap A \cap B$ , then we are in the situation of Corollary 2.6 after an affine-integral coordinate transformation. But the comparison of the monoids is already possible under a weaker assumption, as suggested by the example above and [2, Corollary 5.8], to which we will come back in Corollary 3.7.

**Corollary 2.8.** *Let  $A, B \subset \mathbb{Z}^m$  be finite and suppose that  $\text{aff}(A)$  and  $\text{aff}(B)$  intersect in a single point  $p_0$ . Furthermore suppose  $(kp_0, k) \in M(A) \cap M(B)$  for the smallest  $k > 0$  such that  $kp_0 \in \mathbb{Z}^m$ . Set  $x = (kp_0, k) \oplus 0$  and  $y = 0 \oplus (kp_0, k)$ . Then*

- (1)  $K[M(A \cup B)] \cong K[M(A) \oplus M(B)]/(X^x - X^y)$ ;
- (2)  $\mathbb{H}_{M(A \cup B)} = (1 - T^{(kp_0, k)})\mathbb{H}_{M(A)}\mathbb{H}_{M(B)}$ .

*Proof.* Since  $\text{rank } M(A \cup B) = \text{rank } M(A) + \text{rank } M(B) - 1$  under our hypotheses, we are in the situation of Corollary 2.5, except that the unimodularity of  $x - y$  needs a different argument: it holds since  $(kp_0, k)$  has coprime entries. (Note that we have *not* defined  $k$  by the condition that  $kp_0 \in M(A) \cap M(B)$ .)  $\square$

### 3. NORMALITY

Let  $P \subset \mathbb{R}^m$  be a rational polytope. Then the (ordinary) *Ehrhart function* is given by

$$E(P, k) = \#(kP \cap \mathbb{Z}^d)$$

and the corresponding generating function  $E_P \sum_{k=0}^{\infty} E(P, k)T^k$  is the *Ehrhart series*. In order to interpret the Ehrhart series as a Hilbert series one forms the monoid

$$\mathcal{E}(P) = \{(x, k) : x \in kP \cap \mathbb{Z}^m\} \subset \mathbb{Z}^{m+1}.$$

By Gordan's lemma  $\mathcal{E}(P)$  is an affine monoid, and the Ehrhart series of  $P$  is just the standard Hilbert series of  $\mathcal{E}(P)$ . We define the *multigraded* or *fine Ehrhart series* (or lattice point generating function) of  $P$  by

$$\mathbb{E}_P = \mathbb{H}_{\mathcal{E}(P)}.$$

It is tempting to interpret the results in Section 2 as statements about Ehrhart series. Such an interpretation is indeed possible and will be given below, but it requires further hypotheses. Let us consider the situation of Corollary 2.6 and rational polytopes  $P$  and  $Q$  in  $\mathbb{R}^m$ , such that  $0 \in P \cap Q$ , the vector subspaces  $\mathbb{R}P$  and  $\mathbb{R}Q$  intersect only in 0, and

$$(\mathbb{Z}^m \cap \mathbb{R}R) = (\mathbb{Z}^m \cap \mathbb{R}P) + (\mathbb{Z}^m \cap \mathbb{R}Q).$$

Then we say that  $\text{conv}(P \cup Q)$  is the (convex) *free sum* of  $P$  and  $Q$  (see Henk, Richter-Gebert and Ziegler [12] for further information). The free sum of polytopes can be constructed from the free join by projecting along the line through the representatives of the origins in the free join. Figure 4 illustrates this construction.

We would like to conclude that  $\mathbb{E}_R = (1 - T_{m+1})\mathbb{E}_P\mathbb{E}_Q$ . This conclusion is equivalent to the fact that  $\mathcal{E}(R)$  arises from  $\mathcal{E}(P) \oplus \mathcal{E}(Q)$  via the construction in Corollary 2.5. In general this is not the case, even if the evidently necessary conditions are satisfied.



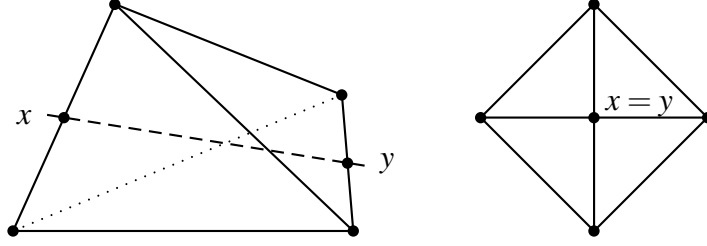


FIGURE 4. From the free join of two line segments to their free sum

**Example 3.1.** Let  $P \subset \mathbb{R}^3$  be the lattice polytope spanned by the points  $-(e_1 + e_2 + e_3)$ ,  $e_i$ ,  $e_i + e_j$ ,  $i, j = 1, 2, 3, i \neq j$  ( $e_i$  denotes the  $i$ -th unit vector). For  $Q$  we choose the interval  $[-1, 2] \subset \mathbb{R}$ . Consider  $P$  and  $Q$  as lattice polytopes in  $\mathbb{R}^4 = \mathbb{R}^3 \oplus \mathbb{R}$ . Then  $R = \text{conv}(P \cup Q)$  is indeed the free sum of  $P$  and  $Q$ .

Set  $A = P \cap \mathbb{Z}^4$  and  $B = Q \cap \mathbb{Z}^4$ . That  $\mathcal{E}(Q) = M(B)$  holds for trivial reasons, and using Normaliz [8] one checks that  $\mathcal{E}(P) = M(A)$ . One even has  $R \cap \mathbb{Z}^4 = A \cup B$ . Nevertheless  $\mathcal{E}(R) \neq M(A \cup B)$ . This can be checked by Normaliz directly or by inspection of the Ehrhart series:

$$E_{P \oplus Q} = \frac{1 + 3T + 5T^2 + 4T^3 + 2T^4}{(1 - T)^5} \neq \frac{1 + 3T + 4T^2 + 5T^3 + 2T^4}{(1 - T)^5} = (1 - T)E_P E_Q.$$

As we will see in Corollary 3.6, this inequality is not a surprise.

In the following we will have to adjoin inverse elements to the affine monoid  $M$ ; see [6, p. 62]. Relative to [6] we use the shortcut  $M[-G] = M[-(G \cap M)]$  for faces  $G$  of  $\text{cone}(M)$ . Extensions of type  $M[-G]$  appear naturally when localizations of monoid domains are to be considered since the subsets  $G \cap M$  of  $M$  are exactly those complementary to prime ideals. The following characterization of regular localizations  $K[M]_P$  is only implicitly given in [6].

**Lemma 3.2.** *Let  $M$  be an affine monoid of rank  $d$ ,  $K$  a field, and  $P$  a prime ideal in  $K[M]$ . Let  $Q$  be the (automatically prime) ideal generated by all monomials  $X^x \in P$  and let  $F$  be the face of  $\text{cone}(M)$  spanned by all  $y \in M$ ,  $X^y \notin Q$ . Then the following are equivalent:*

- (1)  $K[M]_Q$  is a regular local ring;
- (2)  $K[M]_P$  is a regular local ring;
- (3)  $M[-F]$  is isomorphic to  $\mathbb{Z}^{d-n} \oplus \mathbb{Z}_+^n$  for some  $n$ ,  $0 \leq n \leq d$ .

*Proof.* The implications (3)  $\implies$  (2)  $\implies$  (1) hold since regularity is preserved under localizations.

For (1)  $\implies$  (3) it is enough that  $R = K[M[-F]]$  is a regular ring; see [6, 4.45]. This follows from general principles that hold for  $^*$ local rings; see the discussion in [6, p. 208]. Nevertheless a direct argument may be welcome. The crucial observation is that every (prime) ideal of  $R$  generated by monomials is contained in  $Q' = QR$ .

First we show that  $R$  is normal. To this end let  $\bar{R}$  be the normalization of  $R$ . It is itself an affine monoid domain and a finitely generated  $R$ -module. The localization  $(\bar{R}/R)_{Q'}$  vanishes since  $R_{Q'}$  is regular and thus normal. But then  $\bar{R}/R$  vanishes since its support would have to contain a monomial prime ideal if it were empty.

By [6, 4.45] factoriality of  $R$  is sufficient for (3), and it holds if all monomial height 1 prime ideals  $P'$  are principal (Chouinard's theorem [6, 4.56]). But this follows by the same argument that shows normality: a monomial generating the extension of  $P'$  to the factorial ring  $R_{Q'}$  must generate  $P'$  itself.  $\square$

The key to results about Ehrhart series is the preservation of normality in the situation of Theorem 2.1. As we will see, normality depends on the height of monoid elements over facets: every  $x \in M$  has a well-defined (lattice) *height* over a facet  $F$  of  $\text{cone}(M)$ , we denote it by  $\text{ht}_F(x)$ . It is the number of hyperplanes between  $F$  and  $x$  parallel to  $F$  that pass through lattice points and do not contain  $F$ ; so  $\text{ht}_F(x) = 0$  if and only if  $x \in F$ .

**Theorem 3.3.** *Let  $M$  be a normal affine monoid of rank  $d$ , and Suppose that  $x, y \in M$  satisfy conditions (2)(a) and (b) of Theorem 2.1. Then the following are equivalent:*

- (1)  $K[M]/(X^x - X^y)$  is normal.
- (2) If  $G$  is a subfacet of  $\text{cone}(M)$  such that  $x, y \notin G$ , then  $M[-G] \cong \mathbb{Z}^{d-2} \oplus \mathbb{Z}_+^2$ , and  $x$  or  $y$  has height 1 over one of the exactly two facets  $F', F''$  containing  $G$ .

*Proof.* As for Theorem 2.1, we start with the implication (2)  $\implies$  (1). By Hochster's theorem  $K[M]$  is Cohen-Macaulay [6, 6.10], and thus Theorem 2.1 implies that  $K[M]/(X^x - X^y)$  is Cohen-Macaulay. Moreover,  $K[M]/(X^x - X^y) \cong K[M']$  where  $M'$  is the image of  $M$  in  $\text{gp}(M)/\mathbb{Z}(x - y)$ .

It is enough to show that  $K[M']$  satisfies Serre's condition  $(R_1)$  since  $(S_2)$  follows from Cohen-Macaulayness (see [6, 4.F] for Serre's conditions and normality). Since  $K[M']$  is a monoid domain, it is enough to check that the localizations with respect to monomial prime ideals of height 1 are regular [6, Exerc. 4.16]. Let  $P$  be such a prime ideal in  $K[M']$ . The preimage  $Q$  in  $K[M]$  has height 2 and contains  $X^x - X^y$ . There are two cases to distinguish: (i)  $X^x, X^y \notin Q$  and (ii)  $X^x, X^y \in Q$ . In fact,  $Q$  contains either both monomials or none.

Somewhat surprisingly, case (i) does not imply any other condition on  $x$  and  $y$  than those occurring already in Theorem 2.1, which are satisfied by hypothesis. Let  $Q'$  be the ideal generated by all monomials in  $Q$ . We have  $0 \neq Q' \neq Q$  since  $Q'$  contains monomials, but  $X^x, X^y \notin Q$ . Therefore all monomials outside the facet  $F$  of  $\text{cone}(M)$  corresponding to  $Q'$  are inverted in the passage to  $K[M]_Q$ . Since  $M$  is normal,  $M[-F] \cong \mathbb{Z}^{d-1} \oplus \mathbb{Z}_+$ , and  $x$  and  $y$  belong to  $\mathbb{Z}^{d-1}$  because they are not in  $Q'$ . Since  $x - y$  is a basis element in  $\text{gp}(M)$ , it is a basis element of the subgroup  $\mathbb{Z}^{d-1}$ , and  $K[\mathbb{Z}^{d-1} \oplus \mathbb{Z}_+]/(X^x - X^y)$  is a regular ring (see Remark 2.2(c)). Its localization  $K[M']_P$  is therefore also regular.

Now we turn to case (ii). We write the subfacet  $G$  of  $\text{cone}(M)$  corresponding to  $Q$  as the intersection of facets  $F'$  and  $F''$ . Let  $Q'$  and  $Q''$  be the corresponding height 1 prime ideals. Since  $X^x$  and  $X^y$  cannot occur together in  $Q'$  or  $Q''$ , one of them, say  $X^x$ , lies in  $Q'$  and  $X^y$  lies in  $Q''$ . Since  $M[-(F' \cap F'')] \cong \mathbb{Z}^{d-2} \oplus \mathbb{Z}_+^2$ , the localization  $K[M]_Q$  is a regular local ring. Choosing bases in the summands, we write  $K[M[-(F' \cap F'')]] = K[Z_1^{\pm 1}, \dots, Z_{d-2}^{\pm 1}, U, V]$ . In this notation

$$X^x - X^y = \mu U^{\text{ht}_{F'}(x)} - \nu V^{\text{ht}_{F''}(y)}, \quad \mu, \nu \text{ monomials in } K[Z_1^{\pm 1}, \dots, Z_{d-2}^{\pm 1}].$$

The full localization  $K[M]_Q$  is reached if we invert all elements in  $K[Z_1^{\pm 1}, \dots, Z_{d-2}^{\pm 1}, U, V]$  outside the prime ideal generated by  $U$  and  $V$ . The residue class ring modulo  $X^x - X^y$

is regular if (and only if)  $X^x - X^y \in Q_Q \setminus (Q_Q)^2$ , and this is equivalent to  $\text{ht}_F(x) \leq 1$  or  $\text{ht}_G(y) \leq 1$ .

For the converse implication (1)  $\implies$  (2) one has to reverse the arguments just used in the case (ii). First, the regularity of  $K[M']_P = K[M]_Q/(X^x - X^y)$  implies the regularity of  $K[M]_Q$  since the Krull dimension goes up by 1 and the number of generators of the maximal ideal by at most 1. Now Lemma 3.2 gives the structure of  $M[-G]$ . Moreover, as said already,  $K[M']_P = K[M]_Q/(X^x - X^y)$  is regular only if  $X^x - X^y \in Q_Q \setminus (Q_Q)^2$ .  $\square$

We draw consequences similar to those of Theorem 2.1.

**Corollary 3.4.** *Under the hypotheses of Corollary 2.3 the following are equivalent:*

- (1)  $K[M]/(X^{x_1} - X^{x_2}, \dots, X^{x_{n-1}} - X^{x_n})$  is a normal domain;
- (2) for each face  $F$  such that  $\text{rank } M - \dim F = n$  and  $x_1, \dots, x_n \notin F$ , one has the following:
  - (a)  $M[-F] \cong \mathbb{Z}^{d-n} \oplus \mathbb{Z}_+^n$ ;
  - (b) at least  $n-1$  of the  $n$  nonzero numbers  $\text{ht}_{F_i}(x_j)$  are equal to 1 for the facets  $F_1, \dots, F_n$  containing  $F$  and  $j = 1, \dots, n$ .

In particular, it is sufficient for (1) that all  $n$  nonzero heights  $\text{ht}_{F_i}(x_j)$  are equal to 1 in the situation of (2).

*Proof.* The equivalence of (1) and (2) follows by arguments entirely analogous to those proving the theorem, except that the critical localizations are now of type  $M[-F] = \mathbb{Z}^{d-n} \oplus \mathbb{Z}_+^n$ , and the regularity of the residue class ring modulo  $(X^{x_1} - X^{x_2}, \dots, X^{x_{n-1}} - X^{x_n})$  is the crucial condition.

For the last statement we observe that the normal monoid  $M[-G]$  splits into a direct sum of its unit group and a positive (normal) affine monoid of rank  $n$  [6, 2.26]. The positive component must be isomorphic to  $\mathbb{Z}_+^n$ . In fact, the standard map [6, p. 59] sends it surjectively and therefore isomorphically onto  $\mathbb{Z}_+^n$ .  $\square$

**Corollary 3.5.** *Let  $L, M$  and  $N$  be normal affine monoids,  $\varphi : M \oplus N \rightarrow L$  a surjective homomorphism with  $\text{rank } L = \text{rank } M + \text{rank } N - 1$ , and suppose that  $\varphi(x) = \varphi(y)$  for  $x \in M, y \in N, x \neq 0$  or  $y \neq 0$ . Then the following are equivalent:*

- (1)  $K[L] = K[M \oplus N]/(X^x - X^y)$  and  $K[L]$  is normal,
- (2)  $\text{ht}_F(x) \leq 1$  for all facets  $F$  of  $\text{cone}(M)$  or  $\text{ht}_G(y) \leq 1$  for all facets  $G$  of  $\text{cone}(N)$ .

*Proof.* If (2) is satisfied, then  $x - y$  is unimodular in  $\text{gp}(M \oplus N)$ , and we need no longer think about the isomorphism  $K[L] = K[M \oplus N]/(X^x - X^y)$ .

In checking the equivalence of (1) and (2) in regard to normality, one notes that the critical subfacets of  $\text{cone}(M \oplus N)$  are exactly the intersections  $F' \cap F''$  where  $F'$  is the extension of a facet of  $\text{cone}(M)$  not containing  $x$  and  $F''$  extends a facet of  $\text{cone}(N)$  not containing  $y$ , and all such pairs  $(F', F'')$  must be considered.  $\square$

We want to state consequences for Ehrhart series similar to Corollaries 2.6 and 2.8. In the situation of the free sum (and similarly in that analogous to Corollary 2.8) one always has a homomorphism  $\varphi : \mathcal{E}(P) \oplus \mathcal{E}(Q) \rightarrow \mathcal{E}(R)$  where  $R = \text{conv}(P \cup Q)$ . Set  $L = \text{Im } \varphi$ . By Corollary 2.5 we have  $\mathbb{H}_L = (1 - T_{m+1})\mathbb{E}_P\mathbb{E}_Q$ . But  $L$  and  $\mathcal{E}(R)$  generate the same cone in  $\mathbb{R}^{m+1}$  (since  $R = \text{conv}(P \cap Q)$ ) and the same subgroup of  $\mathbb{Z}^{m+1}$  (since  $(\mathbb{Z}^m \cap \mathbb{R}R) =$

$(\mathbb{Z}^m \cap \mathbb{R}P) + (\mathbb{Z}^m \cap \mathbb{R}Q)$ , and  $\mathcal{E}(R)$  is normal. Therefore  $\mathcal{E}(R)$  is the normalization of  $L$ , and the following statements are equivalent: (i)  $L$  is normal, (ii)  $L = \mathcal{E}(R)$ , and (iii)  $\mathbb{H}_L = \mathbb{E}_R$ .

After these preparations we obtain [2, Theorem 1.3]. It generalizes [5, Corollary 1] properly (see [2, Remark 3.5]).

**Corollary 3.6.** *Let  $R \subset \mathbb{R}^m$  be a rational polytope that is the free sum of the rational polytopes  $P$  and  $Q$ , both containing 0. Then the following are equivalent:*

- (1) *At least in one of  $P$  or  $Q$  the origin has height  $\leq 1$  over all facets;*
- (2)  $\mathbb{E}_R = (1 - T_{m+1})\mathbb{E}_P\mathbb{E}_Q$ .

In the same way, as Corollary 2.8 generalizes Corollary 2.6, we can generalize Corollary 3.6 and thus generalize [2, Corollary 5.8], but we must also generalize the condition  $(\mathbb{Z}^m \cap \mathbb{R}R) = (\mathbb{Z}^m \cap \mathbb{R}P) \oplus (\mathbb{Z}^m \cap \mathbb{R}Q)$ . To this end we say that a subset  $A$  of  $\mathbb{Z}^m$  is the  $\mathbb{Z}$ -affine hull of  $B \subset \mathbb{Z}^m$  if

$$A = \{a_1x_1 + \cdots + a_nx_n : n \geq 1, x_1, \dots, x_n \in B, a_1, \dots, a_n \in \mathbb{Z}, a_1 + \cdots + a_n = 1\}.$$

Note that the  $\mathbb{Z}$ -affine hull is the subgroup generated by  $B$  if  $0 \in B$ .

**Corollary 3.7.** *Let  $P, Q \subset \mathbb{R}^m$  be rational polytopes such that  $\text{aff}(P)$  and  $\text{aff}(Q)$  meet in a single point  $p_0 \in P \cap Q$ . Set  $R = \text{conv}(P \cup Q)$  and suppose that  $\text{aff}(R) \cap \mathbb{Z}^m$  is the  $\mathbb{Z}$ -affine hull of  $(\text{aff}(P) \cup \text{aff}(Q)) \cap \mathbb{Z}^m$ . Furthermore let  $k$  be the smallest positive integer such that  $kp_0 \in \mathbb{Z}^m$ . Then the following are equivalent:*

- (1) *At least in one of  $\mathcal{E}(P)$  or  $\mathcal{E}(Q)$  the point  $(kp_0, k)$  has height  $\leq 1$  over all facets;*
- (2)  $\mathbb{E}_R = (1 - T^{(kp_0, k)})\mathbb{E}_P\mathbb{E}_Q$ .

Finally we derive [9, Theorem 3] without using arguments on triangulations.

**Corollary 3.8.** *Let  $M$  be an affine monoid such that  $K[M]$  is Gorenstein and let  $X^w$ ,  $w \in M$ , generate the canonical module of  $K[M]$ . Furthermore let  $x_1, \dots, x_n \in M$  noninvertible elements such that  $w = x_1 + \cdots + x_n$ . Then  $K[M]/(X^{x_1} - X^{x_2}, \dots, X^{x_{n-1}} - X^{x_n})$  is again a Gorenstein normal affine monoid domain and has dimension  $\text{rank } M - (n - 1)$ .*

*Proof.* The point  $w$  is distinguished by the fact that it has height 1 over each facet. Therefore “height vectors” defined  $x_1, \dots, x_n$  are 0-1-vectors with disjoint supports, and Corollary 3.4 applies. It yields that the residue class ring is a normal affine monoid domain, and the Gorenstein property is preserved modulo regular sequences.  $\square$

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